STABILITY IN THE CRITICAL CASE OF THREE PAIRS OF PURE IMAGINARY ROOTS IN THE PRESENCE OF INTERNAL RESONANCE

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Stability of the trivial solution of an autonomous system is investigated for the critical case of three pairs of pure imaginary roots when the ratios of the frequencies of the linearized system are integral. Necessary and sufficient conditions for the stability in the first nonlinear order are given. The results obtained are extended to embrace the degenerate case, the latter including the Hamiltonian systems.

We consider the following system of differential equations:

$$x^{\cdot}_s = \sum_{r=1}^{6} p_{sr} x_r + X_s(x_1, \ldots, x_6) \qquad (s = 1, \ldots, 6) \tag{1}$$

where X_s are holomorphic functions whose expansions in the powers of $x_1, \ldots, x_6$ begin with the terms of at least second order and the constants p_{sr} are such that the matrix $\{p_{sr}\}$ has only pure imaginary eigenvalues $\pm i\lambda_s$, $(s = 1,2,3)$. According to [1-3] the $(k+1)$-th order internal resonance exists in the system (1), if the frequencies $\lambda_1, \lambda_2, \lambda_3$ satisfy relationsgips of the type

$$k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0, \qquad |k_1| + |k_2| + |k_3| = k + 1$$

where k_s are any integers and k is the lowest exponent of the forms appearing at the beginning at the expansions of X_s. The third order resonance of the form $\lambda_3 = \lambda_2 + \lambda_1$ was studied in [3] where, except degenerate cases (*), the necessary and sufficient conditions of conservation of neutrality in the second order were obtained. We consider a more general problem, assuming for definiteness that the frequencies λ_1, λ_2 and λ_3 satisfy the relation

$$p_3\lambda_3 = p_2\lambda_2 + p_1\lambda_1 \tag{2}$$

where p_1, p_2, p_3 are positive integers whose sum is $k+1$.

We know that a special linear transformation yields (1) in the form

$$z^{\cdot}_s = i\lambda_s z_s + Z_s, \qquad \bar{z}^{\cdot}_s = -i\lambda_s \bar{z}_s + \bar{Z}_s \qquad (s = 1, 2, 3) \tag{3}$$

where z_s, $\bar{z}_s$ are complex conjugate variables and the complex conjugate functions $Z_s, \bar{Z}_s$ appear in the form of absolutely convergent series in increasing powers of $z_1, \ldots, z_3, \bar{z}_1, \ldots, \bar{z}_3$, beginning with forms of the k-th order.

We use the following formulas to pass from z_s, $\bar{z}_s$, to the new variables σ_s, $\bar{\sigma}_s$:

$$z_s = [\sigma_s - \Sigma u_s^* \sigma_1^{k_{s1}} \ldots \sigma_3^{k_{s3}} \bar{\sigma}_1^{l_{s1}} \ldots \bar{\sigma}_3^{l_{s3}}] \exp(i\lambda_s t)$$

$$\bar{z}_s = [\bar{\sigma}_s - \Sigma \bar{u}_s^* \bar{\sigma}_1^{k_{s1}} \ldots \bar{\sigma}_3^{k_{s3}} \sigma_1^{l_{s1}} \ldots \sigma_3^{l_{s3}}] \exp(-i\lambda_s t)$$

Here u_s^*, $\bar{u}_s^*$ are periodic functions of time t, to be determined (the asterisk replaces

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*) It is shown below, in particular, that any Hamiltonian system is a completely degenerate case.

the index $(k_{s1}, \ldots, k_{s3}, l_{s1}, \ldots, l_{s3})$) and summation is performed over all nonnegative integers $k_{s1}, \ldots, k_{s3}, l_{s1}, \ldots, l_{s3}$ whose sum is equal to k. The first group of the conjugate equations in the new variables becomes

$$\sigma^{\cdot}_s = \Sigma\,[R_s^* \exp(i\varkappa_s^* t) + u_s^{*\cdot}]\, \sigma_1^{k_{s1}} \ldots \sigma_3^{k_{s3}}\, \bar{\sigma}_1^{l_{s1}} \ldots \bar{\sigma}_3^{l_{s3}} + \Phi_s \tag{4}$$
$$\varkappa_s^* = -\lambda_s + \lambda_1 (k_{s1} - l_{s1}) + \ldots + \lambda_3 (k_{s3} - l_{s3}) \qquad (s = 1, 2, 3)$$

Here R_s^* are complex coefficients of the kth order forms which begin the expansions of Z_s in (3), and Φ_s represent the series in $\sigma_1, \ldots, \sigma_3, \bar{\sigma}_1, \ldots, \bar{\sigma}_3$ beginning with the terms of at least ($k+1$)-th order, the coefficients of these series being periodic in time.

Functions u_s^* are obtained from

$$u_s^{*\cdot} + R_s^* \exp(i\varkappa_s^* t) = P_s^*$$

where P_s^* are constants which must vanish when $\varkappa_s^* \neq 0$ (in order to make u_s^* periodic) and become $P_s^* = R_s^*$ for $\varkappa_s^* = 0$. When the values of k are even, the relations

$$\lambda_s = (k_{s1} - l_{s1})\,\lambda_1 + \ldots + (k_{s3} - l_{s3})\lambda_3$$
$$k_{s1} + l_{s1} + \ldots + k_{s3} + l_{s3} = k \qquad (s = 1, 2, 3) \tag{5}$$

are satisfied, provided that condition (2) holds, by a unique set of numbers k_{sj} and l_{sj} given by

$$l_{11} = p_1 - 1, \quad l_{12} = p_2, \quad k_{13} = p_3, \quad k_{11} = k_{12} = l_{13} = 0$$
$$l_{21} = p_1, \quad l_{22} = p_2 - 1, \quad k_{23} = p_3, \quad k_{21} = k_{22} = l_{23} = 0$$
$$k_{31} = p_1, \quad k_{32} = p_2, \quad l_{33} = p_3 - 1, \quad l_{31} = l_{32} = l_{33} = 0$$

Assuming that P_s^*, for which relations (5) hold, are equal to $a_s + ib_s$ and transforming to the polar coordinates r_s, θ_s according to the formulas

$$\sigma_s = r_s \exp(i\theta_s), \qquad \bar{\sigma}_s = r_s \exp(-i\theta_s)$$

we obtain the following differential equations:

$$r_s^{\cdot} = r_1^{p_1} \ldots r_s^{p_s - 1} \ldots r_3^{p_3} [a_s \cos\theta + (-1)^\delta\, b_s \sin\theta] + \ldots$$
$$r_s\theta_s^{\cdot} = r_1^{p_1} \ldots r_s^{p_s - 1} \ldots r_3^{p_3} [b_s \cos\theta - (-1)^\delta\, a_s \sin\theta] + \ldots \tag{6}$$
$$\theta = p_3\theta_3 - p_2\theta_2 - p_1\theta_1$$
$$\delta = 1 \ \text{for}\ s = 1,2, \quad \delta = 0 \ \text{for}\ s = 3$$

where the terms which have not been written out in full are of at least ($k+1$)-th order in the variables r_1, r_2, r_3. It is the behavior of these variables that provides the solution of the problem of stability. The following theorem holds for the system (6).

Theorem. The trivial solution of the system (6) is stable if and only if no change of sign occurs in the numerical series

$$\beta_1 = -(a_2 b_3 + a_3 b_2), \quad \beta_2 = (a_1 b_3 + a_3 b_1), \quad \beta_3 = a_1 b_2 - a_2 b_1$$

It can easily be checked by direct substitution that the system (6) has, with the accuracy of up to the kth order terms, the following first integral:

$$\beta_1 r_1^2 + \beta_2 r_2^2 + \beta_3 r_3^2 = \text{const}$$

which is sign-definite when no change of sign occurs in the series of numbers $\beta_1, \beta_2, \beta_3$, and this proves the sufficiency of the Theorem condition ; the necessity is proved with the help of Chetaev theorem on instability [4]. The Chetaev function is taken in the form

$$V = r_1^{p_1} r_2^{p_2} r_3^{p_3} (\cos\theta + \varkappa \sin\theta) \tag{7}$$

where $\varkappa$ is a constant to be chosen as convenient. By Eqs.(6) the derivative $V^{\cdot}$ is given, with the accuracy of up to the kth order terms, by

$$V^{\cdot} = r_1^{2(p_1-1)} r_2^{2(p_2-1)} r_3^{2(p_3-1)} [p_1 r_2^2 r_3^2 (a_1 - \varkappa b_1) + p_2 r_1^2 r_3^2 (a_2 - \varkappa b_2) + p_3 r_1^2 r_2^2 (a_3 + \varkappa b_3)]$$

Let us for example assume that

$$\beta_1 < 0, \quad \beta_2 > 0, \quad \beta_3 > 0 \tag{8}$$

Setting $\varkappa = a_2 / b_2$ we obtain

$$V^{\cdot} = r_1^{2(p_1-1)} r_2^{2(p_2-1)} r_3^{2(p_3-1)} [p_1 r_2^2 r_3^2 (a_1 b_2 - a_2 b_1) + p_3 r_1^2 r_2^2 (a_3 b_2 + a_2 b_3)] \frac{1}{b_2} \tag{9}$$

from which we see that $V^{\cdot}$ does not change its sign in the region $r_1 > 0,\ r_2 > 0,\ r_3 > 0$ By suitable selection of the range of variation of θ one can ensure that both, V and its derivative have the same sign. Thus, in some region $r_s > 0$ $(s = 1,2,3)$ we have $\theta' \leqslant \theta \leqslant \theta''$ and $V(\theta') = V(\theta'') = 0$ and the derivative $V^{\cdot}$ will be sign-definite and of the same sign as V with the accuracy of up to the kth order terms. Consequently, on the basis of the Chetaev theorem the trivial solution $r_s = 0$ $(s = 1,2,3)$ is unstable. For $a_2 = 0$ we can set $\varkappa = a_3 / b_3$ and repeat the previous arguments (the case of $\beta_s = 0$ will be dealt with below).

Similar arguments are valid for any combination of signs of numbers $\beta_1, \beta_2, \beta_3$.

The degenerate case. We consider a particular case when at least one of the quantities $\beta_1, \beta_2, \beta_3$ becomes zero. To start with we assume that β_1, β_2 and β_3 do not vanish simultaneously. Then the trivial solution is unstable no matter what signatures are assigned to those β_s which are not zero. For example, let

$$a_1 b_3 + a_3 b_1 = 0$$

Setting $\varkappa = a_1 / b_1$ we obtain

$$V = \frac{p_2}{b_1} r_1^{2p_1} r_2^{2(p_2-1)} r_3^{2p_3} (a_2 b_1 + a_1 b_2)$$

and if $a_2 b_1 + a_1 b_2 \neq 0$, then $V^{\cdot}$ will be sign-definite in the region $r_s \geqslant 0$, $\theta' \leqslant \theta \leqslant \theta''$. If, on the other hand, $a_2 b_1 + a_1 b_2 = 0$, we can set $\varkappa = a_2 / b_2$, thus proving the instability since $a_3 b_2 + a_2 b_3 \neq 0$.

Let us now consider a particularly degenerate case when $\beta_1, \beta_2, \beta_3$ all vanish, i.e. when

$$a_2 b_3 + a_3 b_2 = a_1 b_3 + a_3 b_1 = a_1 b_2 - a_2 b_1 = 0 \tag{10}$$

Although the latter case is very specific, it is nevertheless of great practical value since these relations are always true for canonical Hamiltonian systems. Indeed, in order that the variables σ_s and $\bar{\sigma}_s$ satisfying (4) and conjugate to them can be canonical, conditions of the form

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \pm \frac{p_1}{p_2}, \quad \frac{a_1}{a_3} = -\frac{b_1}{b_3} = \pm \frac{p_1}{p_3}, \quad \frac{a_2}{a_3} = -\frac{b_2}{b_3} = \pm \frac{p_2}{p_3} \tag{11}$$

must hold, and this is readily checked by direct substitution. The choice of sign is governed by the method of partitioning the variables σ_s and $\bar{\sigma}_s$, $(s = 1,2,3)$ into sets of conjugate canonical variables.

We shall show that in this case the system remains neutral in the kth approximation if and only if a change of sign occurs in the numerical sequence a_1, a_2, a_3 (or $b_1, b_2, -b_3$).

The sufficiency is proved by the existence of the following sign-definite integral:

$$r_1^2 + \alpha_2 r_2^2 + \alpha_3 r_3^2 = \text{const}, \quad \alpha_2 > 0, \quad \alpha_3 > 0 \tag{12}$$

For (12) to be the integral of (6), it is necessary and sufficient that α_2 and α_3 satisfy the equations

$$a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0, \quad b_1 + \alpha_2 b_2 - \alpha_3 b_3 = 0 \tag{13}$$

which by virtue of (11) are identically equal to each other. Discarding the second equation we conclude that an $\alpha_2 > 0$ and an $\alpha_3 > 0$ satisfying (13) can always be found provided that a change of sign occurs in the sequence a_1, a_2, a_3 .

If a_1, a_2, a_3 all have the same sign, then setting $\varkappa = 0$ in (7) and employing the Chetaev theorem we find, that the trivial solution is unstable in the kth approximation which proves the necessity of the above conditions.

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ON THE JUSTIFICATION OF THE PRINCIPLE OF POTENTIAL ENERGY MINIMUM IN PROBLEMS OF EQUILIBRIUM STABILITY OF NONLINEARLY ELASTIC MEMBRANES

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A sufficient indication of the stability of the form of equilibrium is given for an elastic axisymmetric shell, assuming that initial perturbations are axisymmetric. It is also assumed that the energy stored in an element of the shell, is determined only by the

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*) A. M. Slobodkin, Candidate of Physics and Mathematics, Senior Scientific Worker in the VTs Akad. Nauk SSSR, submitted this paper to the Editor on the 24th November 1969.

When critically reviewing the paper, it was found that logical rounding up of this study would be completed if condition (1.1) was expressed directly in terms of the initial data of the problem which are the initial form of equilibrium for the shell r_0 (s) and the loading; this means that it is necessary to specify the conditions (even if only partial ones) of the existence of a solution which would satisfy (1.1).

This advice of the Editor could not, unfortunately, be followed because of the premature death of the author.

The Editor publishes now the paper in its original version and expects that the colleagues of A. M. Slobodkin will fill the gap as a contribution to the memory of their fellow.

The abstract of this paper has been slightly modified.